Boundary-Layer Approximation to Powered-Flight Attitude Transients

HENRY J. KELLEY*
Analytical Mechanics Associates Inc., Jericho, N.Y.

Introduction

POWERED-FLIGHT optimization for a rocket vehicle is examined in the following for a model that includes rigid-body degrees of freedom in boundary-layer approximation. The main result is a performance index for attitude transients, which is the time integral of the dot product of the thrust direction with a desired thrust direction determined by point-mass trajectory optimization. It is intuitively reasonable to maximize the projection of the thrust vector upon the desired thrust direction in this sense, and the similarity to the usual integral square error index, which approximates it for small pointing errors, is interesting.

The boundary-layer approximation is along the lines of singular perturbation theory of ordinary differential equations,[1] which uses the ideas and some of the terminology of fluid dynamics boundary-layer theory, but in its smaller sphere is somewhat more highly developed. The central idea of the approach is reduction in the order of the system of differential equations—the state-Euler equation system in the present instance—as a parameter ϵ tends toward zero. After the reduced-order system is solved, the discrepancies with specified end conditions are resolved by introducing a corrective "boundary-layer" transient, also calculated via a reduced-order system of differential equations. Treatment of an optimal aircraft flight problem decoupled into *three* lower-order problems is sketched in Ref. 2, whereas a general low-order problem of simple form is examined more intensively in Ref. 3.

The main point of the present Note is treatment of rigid-body motions in boundary-layer approximation, i.e., the possibility of separate treatment, with only one-way coupling, with the point-mass trajectory optimization in terms of a performance index inherited from that preceding calculation. It is hoped that the transparent illustration of attitude transients for powered-rocket flight in vacuum may draw interest toward the derivation of performance indices for more complex situations, i.e., nonplanar atmospheric flight vehicular maneuvers.

Analysis

The equations of motion for planar vacuum flight of a rocket vehicle are

$$\epsilon\dot{\omega} = \eta/I, \; \epsilon\dot{\theta} = \omega \tag{1}$$

$$\dot{u} = (T/m)\sin\theta + Y, \; \dot{v} = (T/m)\cos\theta + X \tag{2}$$

$$\dot{y} = u, \; \dot{x} = v \tag{3}$$

where u, v are velocity components, y, x position components, T thrust magnitude, m mass, θ thrust attitude angle, Y, X gravitational force components, ω angular rate, I moment of inertia, η control torque, and ϵ a perturbation parameter taking the value zero for the point-mass model and the value unity for the complete model.

The Euler equations for optimization of the complete model are

$$\epsilon\dot{\lambda}_\omega = -\lambda_\theta \tag{4}$$

$$\epsilon\dot{\lambda}_\theta = -(T/m)(\lambda_u\cos\theta - \lambda_v\sin\theta) \tag{5}$$

$$\dot{\lambda}_u = -\lambda_y, \; \dot{\lambda}_v = -\lambda_x \tag{6}$$

$$\dot{\lambda}_y = -\lambda_u Y_y - \lambda_v X_y \tag{7}$$

$$\dot{\lambda}_x = -\lambda_u Y_x - \lambda_v X_x \tag{8}$$

$$\eta = \arg\max_{|\eta| \le \bar{\eta}} (\lambda_\omega \eta/I) \tag{9}$$

where, for simplicity, corrective torque η has been taken bounded and cost-free as far as propellant consumption is concerned.

If the attitude control motions are essentially fast transients superimposed on a slowly and smoothly-varying motion of the center of mass, the optimization of the trajectory can be done in good approximation with the point-mass model, $\epsilon = 0$, and the attitude motion treated as a boundary-layer correction.[1,2,3] To this end, the state and Euler equations are transformed to a new independent variable $\tau = t - t_0/\epsilon$, a "stretched" time scale, for analysis of the transition from given initial attitude and attitude rate:

$$\frac{d\omega^*}{d\tau} = \frac{\eta^*}{I}, \; \frac{d\theta^*}{d\tau} = \omega^*, \; \frac{d\lambda_\omega^*}{d\tau} = -\lambda_\theta^* \tag{10}$$

$$(d/d\tau)\lambda_\theta^* = -(T/\bar{m}_0)(\bar{\lambda}_{u0}\cos\theta^* - \bar{\lambda}_{v0}\sin\theta^*) \tag{11}$$

$$\eta^* = \arg\max_{|\eta| \le \bar{\eta}} (\lambda_\omega^*\eta)/I \tag{12}$$

The optimal point-mass trajectory, denoted by superscribed bars, has the familiar bilinear tangent steering solution for a uniform gravity model, requires numerical optimization for the inverse square gravity law, but, in any case, has been extensively investigated.

The Euler equations for the boundary-layer transient, denoted by superscribed asterisks, are of low order, being decoupled from the point-mass motion, except as the initial values (denoted by subscript zero) enter the right members. These are the Euler equations for the rigid-body motion alone, with the performance index

$$\frac{T}{\bar{m}_0}\int_0^\infty (\bar{\lambda}_{u0}\sin\theta^* + \bar{\lambda}_{v0}\cos\theta^*)d\tau \tag{13}$$

so that one may characterize the transient as one maximizing the time integral of the dot product of the attitude direction vector with the desired thrust direction (or "primer vector").

The result stands even if several complicating factors are brought in, which were purposely omitted for clarity. For example, the point-mass rocket model may be three-dimensional and throttleable, and have any performance index that fits the Mayer mold; the attitude control may be more complex and possess an additional propellant constraint, etc.

The approximation is good to the extent that the time scales of the two motions are really separate. Higher-order corrections to both the reduced and boundary-layer approximations are possible by a systematic process, due to Vasil'eva,[1] similar to the "asymptotic matching" of fluid mechanics boundary-layer theory. Certainly, however, the initial appeal of this approach to attitude dynamics is the possibility for deriving performance indices for rigid-body motions with more complex models. It is of interest that, in the simple case analyzed, the performance index derived provides a rationale for the use of integral squared error, which approximates it for small pointing errors.

References

[1] Wasow, W., "Asymptotic Expansions for Ordinary Differential Equations," *Interscience*, New York, 1965.

[2] Kelley, H. J., "Flight Path Optimization With Multiple Time Scales," *Journal of Aircraft*, to appear.

[3] Kelley, H. J., "Singular Perturbations for a Mayer Variational Problem," *AIAA Journal*, Vol. 8, No. 7, 1970, pp. 1177–1178.

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* Vice President. Associate Fellow, AIAA.